

Emergent quantum Euler equation and Bose-Einstein condensates

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Abstract

In this paper, proceeding from the recently developed way of deriving the quantum-mechanical equations from the classical ones, the complete system of hydrodynamical equations, including the quantum Euler equation, is derived for a perfect fluid and an imperfect fluid with pairwise interaction between the particles. For the Bose-Einstein condensate of the latter one the Bogolyubov spectrum of elementary excitations is easily reproduced in the acoustic approximation.

Keywords: de Broglie-Bohm theory, quantum hydrodynamics, Euler equation, Bose-Einstein condensate, Gross-Pitaevskii equation, superfluid helium

Introduction

Scientific interest in the de Broglie-Bohm causal interpretation of quantum mechanics [1, 2] and its applications has been recently appreciably rekindled (see also, for example, [3]). In particular, in [4], following the ideas of N.G. Chetaev (the theorem on stable trajectories in dynamics [5]) and G. 't Hooft (the classical deterministic theory, supplemented with the mechanism of dissipation, generates the observed quantum behaviour of our world [6]), it was explicitly demonstrated, that adding the dissipation energy Q to the Hamilton function and imposing the stability condition on motion of the corresponding mechanical system, one can come to the standard Schrödinger equation, as well as identify the famous Bohmian potential exactly with the introduced dissipation energy. Besides, the indisputable advantage of the produced derivation of the Schrödinger equation from Newtonian mechanics lies in the fact, that $|\psi|^2$ turns out to represent the density of trajectories in the configuration space.

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One of the next natural steps lies in developing the hydrodynamical approach to the problem of motion and deriving the quantum Euler equation, and then applying it for describing the Bose-Einstein condensates. This paper is devoted exactly to this very important and promising line of investigation. Obviously, if fundamental properties of quantum fluids are theoretically reproduced in the Bohmian mechanics in a natural way without any additional assumptions and ungrounded extensions, than this fact may serve as one more circumstantial evidence of this theory.

The paper is organized in the following way. In Section 1 we derive the modified Liouville equation, taking into consideration quantum effects. Then we come to the complete system of the quantum hydrodynamics equations in Section 2. Finally, in Section 3 we consider sound waves in a quantum imperfect fluid with pairwise interaction between the particles and reproduce the Bogolyubov spectrum of elementary excitations for its Bose-Einstein condensate. The main results are summarized in Conclusion.

1. Modified Liouville equation in de Broglie-Bohm theory

One of the most logical and self-consistent methods for deriving the macroscopic hydrodynamical equations consists in averaging of the microscopic mechanical equations of motion of each single particle. Following this prevalent in fluid physics method, let us consider a system of N identical particles, whose motion obeys Newton's second law:

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad m \frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i(\mathbf{r}_k, t), \quad \mathbf{F}_i(\mathbf{r}_k, t) = - \left. \frac{\partial U}{\partial \mathbf{r}_i} \right|_t, \quad (1)$$

where $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t)$ are the radius-vector and the velocity of the i -th particle respectively, m is the mass of a single particle, \mathbf{F}_i is the force, acting on the i -th particle from the direction of the external field (for example, the gravitational one) and all other particles, and $U(\mathbf{r}_k, t)$ is the corresponding potential energy. For simplicity we restrict ourselves in this paper to the case of uncharged particles in the absent external electromagnetic field¹. Here and in what follows the index k corresponds to the set of natural numbers from 1 to N inclusive.

In order to proceed from the microscopic equations (1) to the macroscopic ones, one needs the N -particle distribution function $f_N(\mathbf{r}_k, \mathbf{v}_k, t)$, satisfying the standard normalization requirement

$$\int f_N(\mathbf{r}_k, \mathbf{v}_k, t) \prod_{i=1}^N d\mathbf{r}_i d\mathbf{v}_i = 1 \quad (2)$$

¹The more general case of charged particles in the present external electromagnetic field is a subject of our forthcoming paper.

and the corresponding well-known continuity equation

$$\frac{df_N}{dt} = \frac{\partial f_N}{\partial t} \Big|_{\mathbf{r}_k, \mathbf{v}_k} + \sum_{i=1}^N \mathbf{v}_i \frac{\partial f_N}{\partial \mathbf{r}_i} \Big|_{\mathbf{v}_k, t} + \frac{1}{m} \sum_{i=1}^N \mathbf{F}_i(\mathbf{r}_k, t) \frac{\partial f_N}{\partial \mathbf{v}_i} \Big|_{\mathbf{r}_k, t} = 0 \quad (3)$$

in the phase space. Along with (2) the function $f_N(\mathbf{r}_k, \mathbf{v}_k, t)$, representing the probability density in the phase space, should satisfy the following evident equation:

$$\int f_N(\mathbf{r}_k, \mathbf{v}_k, t) \prod_{i=1}^N d\mathbf{v}_i = A^2(\mathbf{r}_k, t), \quad (4)$$

where the function $A(\mathbf{r}_k, t) = |\psi(\mathbf{r}_k, t)|$ is the modulus (amplitude) of the N -particle (total) wave function $\psi(\mathbf{r}_k, t)$, obeying the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} \Big|_{\mathbf{r}_k} = -\frac{\hbar^2}{2m} \sum_{i=1}^N \Delta_i \psi + U(\mathbf{r}_k, t) \psi, \quad \Delta_i = \left(\frac{\partial}{\partial \mathbf{r}_i} \Big|_t \right)^2. \quad (5)$$

Obviously, as it directly follows from (2) and (4), for the functions $A(\mathbf{r}_k, t)$ and $\psi(\mathbf{r}_k, t)$ the standard normalization requirement

$$\int A^2(\mathbf{r}_k, t) \prod_{i=1}^N d\mathbf{r}_i = \int |\psi(\mathbf{r}_k, t)|^2 \prod_{i=1}^N d\mathbf{r}_i = 1 \quad (6)$$

holds true (naturally, $A^2(\mathbf{r}_k, t) = |\psi(\mathbf{r}_k, t)|^2$ represents the probability density in the configuration space). Let us also note, that the functions $f_N(\mathbf{r}_k, \mathbf{v}_k, t)$ and $A(\mathbf{r}_k, t)$ are invariant relative to permutation of two arbitrary particles in view of their identity.

According to the de Broglie-Bohm theory [1, 2, 3, 4], the potential energy $U(\mathbf{r}_k, t)$ may be presented in the form

$$U(\mathbf{r}_k, t) = U_{\text{cl}}(\mathbf{r}_k, t) + Q(\mathbf{r}_k, t), \quad (7)$$

where $U_{\text{cl}}(\mathbf{r}_k, t)$ is the classical part, and $Q(\mathbf{r}_k, t)$ is the Bohmian quantum potential:

$$Q(\mathbf{r}_k, t) = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\Delta_j A}{A}. \quad (8)$$

Expressing $A(\mathbf{r}_k, t)$ from (4) and substituting the result into (8), we obtain

$$Q(\mathbf{r}_k, t) = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\Delta_j \sqrt{\int f_N(\mathbf{r}_k, \mathbf{v}_k, t) \prod_{l=1}^N d\mathbf{v}_l}}{\sqrt{\int f_N(\mathbf{r}_k, \mathbf{v}_k, t) \prod_{l=1}^N d\mathbf{v}_l}}. \quad (9)$$

Substituting (1), (7) and (9) into (3), we come to the modified Liouville equation

$$\begin{aligned} & \left. \frac{\partial f_N}{\partial t} \right|_{\mathbf{r}_k, \mathbf{v}_k} + \sum_{i=1}^N \mathbf{v}_i \left. \frac{\partial f_N}{\partial \mathbf{r}_i} \right|_{\mathbf{v}_k, t} - \frac{1}{m} \sum_{i=1}^N \left. \frac{\partial U_{\text{cl}}}{\partial \mathbf{r}_i} \right|_t \left. \frac{\partial f_N}{\partial \mathbf{v}_i} \right|_{\mathbf{r}_k, t} + \\ & + \frac{\hbar^2}{2m^2} \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_i} \left(\sum_{j=1}^N \frac{\Delta_j \sqrt{\int f_N(\mathbf{r}_k, \mathbf{v}_k, t) \prod_{l=1}^N d\mathbf{v}_l}}{\sqrt{\int f_N(\mathbf{r}_k, \mathbf{v}_k, t) \prod_{l=1}^N d\mathbf{v}_l}} \right) \left. \frac{\partial f_N}{\partial \mathbf{v}_i} \right|_{\mathbf{r}_k, t} = 0. \end{aligned} \quad (10)$$

Obviously, the modification lies in the last term of quantum nature, proportional to \hbar^2 . In contrast to three previous terms, it is essentially nonlinear concerning $f_N(\mathbf{r}_k, \mathbf{v}_k, t)$. It should be also mentioned, that the obtained equation (10) contains all information about the system. In other words, strictly speaking, it is not necessary to solve any other additional equation, for example, the Schrödinger equation (5), for solving (10), where all quantum features are already taken into account.

On the other hand, the modified Liouville equation (10) is very difficult to solve, so one needs to carry out some simplifications. The clearest one of them is considered in the next section.

2. Equations of Bohmian hydrodynamics for a perfect fluid

Let us give concrete expression to the classical potential energy $U_{\text{cl}}(\mathbf{r}_k, t)$, neglecting pairwise interaction between the particles (or, in other words, regarding our "fluid" as a perfect one):

$$U_{\text{cl}}(\mathbf{r}_k, t) = \sum_{j=1}^N V(\mathbf{r}_j, t), \quad (11)$$

where $V(\mathbf{r}_j, t)$ is the potential energy of the j -th particle in the external field. For simplicity let us assume, that all N particles are "in the same quantum state", described by the normalized to unity wave function ψ_0 with the modulus A_0 , then the solution of the Schrödinger equation (5) may be presented in the form

$$\psi(\mathbf{r}_k, t) = \prod_{j=1}^N \psi_0(\mathbf{r}_j, t), \quad i\hbar \frac{\partial \psi_0}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}, t) \right) \psi_0, \quad (12)$$

whence it follows, in particular, that

$$A(\mathbf{r}_k, t) = \prod_{j=1}^N A_0(\mathbf{r}_j, t). \quad (13)$$

It should be mentioned, that the function $A_0^2(\mathbf{r}, t)$ represents the macroscopic density of an arbitrary single particle. Consequently, the macroscopic density $n(\mathbf{r}, t)$ of the whole system is interconnected with $A_0^2(\mathbf{r}, t)$ as follows:

$$n(\mathbf{r}, t) = N A_0^2(\mathbf{r}, t) \quad \Leftrightarrow \quad A_0(\mathbf{r}, t) = \sqrt{\frac{n(\mathbf{r}, t)}{N}}. \quad (14)$$

Substituting (13) and (14) into (8), we get

$$Q(\mathbf{r}_k, t) = \sum_{j=1}^N Q_1(\mathbf{r}_j, t), \quad Q_1(\mathbf{r}_j, t) = -\frac{\hbar^2}{2m} \frac{\Delta_j A_0(\mathbf{r}_j, t)}{A_0(\mathbf{r}_j, t)} = -\frac{\hbar^2}{2m} \frac{\Delta_j \sqrt{n(\mathbf{r}_j, t)}}{\sqrt{n(\mathbf{r}_j, t)}}. \quad (15)$$

Let us note, that the made assumption of the same quantum state for all N particles enables to present the N -particle quantum potential $Q(\mathbf{r}_k, t)$ in the form of the sum of N identical 1-particle ones $Q_1(\mathbf{r}_j, t)$, each depending on the radius-vector \mathbf{r}_j of the corresponding j -th particle, $j = 1, 2, \dots, N$. Simultaneously the N -particle amplitude $A(\mathbf{r}_k, t)$ is presented in the form of the product of N identical 1-particle ones $A_0(\mathbf{r}_j, t)$. A similar presentation holds true for the N -particle distribution function:

$$f_N(\mathbf{r}_k, \mathbf{v}_k, t) = \prod_{j=1}^N f_1(\mathbf{r}_j, \mathbf{v}_j, t), \quad (16)$$

where the 1-particle distribution function $f_1(\mathbf{r}, \mathbf{v}, t)$ satisfies the normalization condition

$$\int f_1(\mathbf{r}, \mathbf{v}, t) d\mathbf{r} d\mathbf{v} = 1. \quad (17)$$

As it directly follows from (10) after the substitution of (11) and (16) (or from (3) after the substitution of (1), (7), (11) and (15)), this function obeys the equation

$$\left. \frac{\partial f_1}{\partial t} \right|_{\mathbf{r}, \mathbf{v}} + \mathbf{v} \left. \frac{\partial f_1}{\partial \mathbf{r}} \right|_{\mathbf{v}, t} - \frac{1}{m} \frac{\partial}{\partial \mathbf{r}} \left(V(\mathbf{r}, t) - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{n(\mathbf{r}, t)}}{\sqrt{n(\mathbf{r}, t)}} \right) \left. \frac{\partial f_1}{\partial \mathbf{v}} \right|_{\mathbf{r}, t} = 0. \quad (18)$$

Applying the standard methods of fluid physics (in particular, presenting \mathbf{v} in the form of the sum of the hydrodynamical velocity field $\mathbf{u}(\mathbf{r}, t)$ and the deviation $\delta \mathbf{v}$ from it), one can easily obtain from (18) the complete system of hydrodynamical equations, describing motion of our perfect fluid:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{r}} (\rho \mathbf{u}) = 0, \quad (19)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \frac{\partial}{\partial \mathbf{r}} \right) \mathbf{u} = -\frac{1}{\rho} \frac{\partial p}{\partial \mathbf{r}} - \frac{1}{m} \frac{\partial V}{\partial \mathbf{r}} + \frac{\hbar^2}{2m^2} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), \quad (20)$$

$$\frac{3}{2} \frac{\rho \kappa}{m} \left[\frac{\partial T}{\partial t} + \frac{\partial}{\partial \mathbf{r}} (T \mathbf{u}) \right] = -\frac{\partial \mathbf{q}}{\partial \mathbf{r}}, \quad (21)$$

where the mass density $\rho(\mathbf{r}, t)$, the pressure $p(\mathbf{r}, t)$, the temperature $T(\mathbf{r}, t)$ and, finally, the heat flux density $\mathbf{q}(\mathbf{r}, t)$ are the macroscopic quantities, given by the following equations respectively:

$$\rho(\mathbf{r}, t) = mn(\mathbf{r}, t) = Nm \int f_1(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \quad (22)$$

$$p(\mathbf{r}, t) = n(\mathbf{r}, t) \kappa T(\mathbf{r}, t) = \frac{1}{3} Nm \int \delta v^2 f_1(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \quad (23)$$

$$\mathbf{q}(\mathbf{r}, t) = \frac{1}{2} Nm \int \delta v^2 \delta \mathbf{v} f_1(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}. \quad (24)$$

Here κ is the Boltzmann constant, and viscosity is neglected. The continuity equation (19) and the heat equation (21) exactly coincide with the corresponding equations of classical (non-quantum) hydrodynamics [8]. Only the equation (20), which represents the quantum Euler equation, differs from its classical analogue in the presence of the last term in the right hand side, proportional, in particular, to the ratio \hbar^2/m^2 . Besides, it is absolutely obvious, that one should expect the greater absolute value of the quantum potential Q , the sharper the spatial change of the mass density ρ is. If the fluid is incompressible, that is ρ is simply some constant, then $Q = 0$, and the distinction between classical and quantum Euler equations is eliminated. In the general case the additional quantum term in (20) vanishes, if the function $\sqrt{\rho}$ satisfies the Helmholtz equation $\Delta\sqrt{\rho} + \text{const}\sqrt{\rho} = 0$, where const is an arbitrary constant.

Let us estimate, for which typical velocities and distances in some concrete problem this term, being nonzero, plays a vital part. Introducing typical velocity u_0 and distance r_0 , one can immediately compare it with the second term in the left hand side of the same equation. They become quantities of the same order for $u_0 r_0 \sim \hbar/m$. In this simple relationship one can easily recognize the famous Heisenberg's uncertainty principle, as it should be. For example, if it is a question of helium, then the product $u_0 r_0$ should be a quantity of the order $10^{-8} m^2/s$.

Thus, in this section we have derived the equations (19), (20) and (21) of Bohmian hydrodynamics for a perfect fluid. Concluding this section, let us note, that in the framework of standard quantum mechanics the hydrodynamical equations were written down by Landau and Khalatnikov (see, for example, [7]), but they contain quantum-mechanical operators with proper commutation rules and are much more complicated.

3. Sound waves in imperfect fluid with pairwise interaction

According to [8, 9], only for temperatures, close to absolute zero, quantum effects advance to the forefront in properties of fluids, but actually only helium remains liquid down to absolute zero, while all other fluids become solid before quantum. As regards helium, our basic assumption of the same quantum state

for all particles holds true for the Bose-Einstein condensate of its isotope ^4He , being a Bose fluid (see also [10, 11]). Now we are interested in developing quantum acoustics for such quantum fluids, being, generally speaking, imperfect owing to interparticle interaction.

In order to take proper account of pairwise interaction between the particles, one can add in the right hand side of (11) the term

$$V_{PI} = \frac{1}{2} \sum_{j,l=1;j \neq l}^N \Phi(\mathbf{r}_j - \mathbf{r}_l), \quad (25)$$

where $\Phi(\mathbf{r}_j - \mathbf{r}_l)$ is the potential energy of interaction of j -th and l -th particles, and then apply the method of correlation functions for solving (10)², but there is also another way. Even in spite of interparticle interaction, the form of the equations from (13) to (16) may be preserved as a certain approximation, if the "1-particle" Schrödinger equation (12) is modified in the following way:

$$i\hbar \frac{\partial \psi_0}{\partial t} = \left(-\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}, t) + V_{GP}(\mathbf{r}, t) \right) \psi_0, \quad (26)$$

where the additional potential energy

$$V_{GP}(\mathbf{r}, t) = \frac{4\pi\hbar^2 a}{m} N A_0^2(\mathbf{r}, t) = \frac{4\pi\hbar^2 a}{m^2} \rho(\mathbf{r}, t), \quad (27)$$

where, in its turn, a is a typical length of boson-boson scattering. The nonlinear 1-particle Schrödinger equation (26) is well-known as the Gross-Pitaevskii equation [12, 13] (see also [10, 11]). Taking into consideration the presence of (27), instead of (20) we get the quantum Euler equation in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \frac{\partial}{\partial \mathbf{r}} \right) \mathbf{u} = -\frac{1}{\rho} \frac{\partial p}{\partial \mathbf{r}} - \frac{1}{m} \frac{\partial V}{\partial \mathbf{r}} - \frac{4\pi\hbar^2 a}{m^3} \frac{\partial \rho}{\partial \mathbf{r}} + \frac{\hbar^2}{2m^2} \frac{\partial}{\partial \mathbf{r}} \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right). \quad (28)$$

At the same time two other hydrodynamical equations (19) and (21) remain unchanged.

Now let us turn to the case $V = 0$ and apply in the equations (19) and (28) the standard acoustic approximation: up to the zero order of smallness $\rho = \rho_0$, $\mathbf{u} = 0$, $p = p_0$ (where ρ_0 and p_0 are some constants, depending neither on time t , nor on spatial coordinates \mathbf{r}), and up to the first order of smallness $\rho = \rho_0 + \rho_1$, $\mathbf{u} = \mathbf{u}_1$, $p = p_0 + p_1$ (where the additional terms ρ_1 , \mathbf{u}_1 and p_1 are some functions of t and \mathbf{r} , as well as the presence of the adiabatic relation $p_1 = c_s^2 \rho_1$ is assumed, where c_s is the sound speed). Then from (19) and (28) up to the first order of smallness we obtain respectively

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial \mathbf{u}_1}{\partial \mathbf{r}} = 0, \quad (29)$$

²Development of this method for the modified Liouville equation in the de Broglie-Bohm theory is also a subject of our forthcoming paper.

$$\frac{\partial \mathbf{u}_1}{\partial t} = -\frac{c_s^2}{\rho_0} \frac{\partial \rho_1}{\partial \mathbf{r}} - \frac{4\pi\hbar^2 a}{m^3} \frac{\partial \rho_1}{\partial \mathbf{r}} + \frac{\hbar^2}{4m^2 \rho_0} \frac{\partial}{\partial \mathbf{r}} (\Delta \rho_1). \quad (30)$$

From (29) and (30) the sound propagation equation follows immediately:

$$\frac{\partial^2 \rho_1}{\partial t^2} - \left(c_s^2 + \frac{4\pi\hbar^2 a \rho_0}{m^3} \right) \Delta \rho_1 + \frac{\hbar^2}{4m^2} \Delta (\Delta \rho_1) = 0. \quad (31)$$

Let us establish the dispersion law for plane waves. For that let us look for the solution of (31) in the form

$$\rho_1(\mathbf{r}, t) \sim \exp[i(\mathbf{k}\mathbf{r}) - i\omega t], \quad (32)$$

where \mathbf{k} is a wave vector, and ω is a frequency. Substituting (32) into (31), we get

$$\omega = \sqrt{\left(c_s^2 + \frac{4\pi\hbar^2 a \rho_0}{m^3} \right) k^2 + \frac{\hbar^2 k^4}{4m^2}}, \quad (33)$$

where $k = |\mathbf{k}|$. It should be mentioned, that here the sound speed squared c_s^2 is proportional to temperature and, consequently, becomes negligible for temperatures, close to absolute zero, while the scattering length a depends on temperature weakly, remaining nonzero. Then the established dispersion law (33) exactly coincides with the famous Bogolyubov spectrum of elementary excitations in a Bose fluid. Thus, this fundamental spectrum is derived in our quantum-hydrodynamical approach in a sufficiently simple way.

Conclusion

In this paper, proceeding from the recently developed way of deriving the quantum-mechanical equations from the classical ones [4], we have derived successively the modified Liouville equation (10) and the complete system of hydrodynamical equations (namely, the continuity equation (19), the quantum Euler equation (20) and the heat equation (21)) for a perfect fluid, formed by identical particles, being in the same quantum state. This basic assumption holds true for Bose-Einstein condensates of quantum fluids at very low temperatures. Then we have taken into account the interparticle interaction by using the Gross-Pitaevskii equation (26). Finally, using the acoustic approximation, we have easily reproduced the Bogolyubov spectrum of elementary excitations (see (33)).

In future, being based on the direct relativistic generalization [14] of [4], we shall also generalize the developed here quantum hydrodynamics to the relativistic case. This line of investigation promises to open up new intriguing possibilities for studying, in particular, black holes and the whole Universe.

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